Pentagon Packing Models for "All-Pentamer" Virus Structures

Tibor Tarnai,* Zsolt Gáspár,* and Lídia Szalai‡
*Technical University of Budapest, Department of Civil Engineering Mechanics, Budapest, H-1521; and ‡Budapest, H-1037 Hungary

ABSTRACT A connection is made between 1) the observed structures of virus capsids whose capsomers are all pentamers and 2) the mathematical problem of determination of the largest size of a given number of equal regular spherical pentagons that can be packed on the surface of the unit sphere without overlapping. It is found that papillomaviruses provide the conjectured solution to the spherical pentagon packing problem for 72 pentagons. Thus, a study of some virus structures has given additional insight into a mathematical problem. At the same time this mathematical problem enables prediction of an octahedral form of papillomavirus particles consisting of 24 pentamers. It is also found that the various tubular and spherical "all-pentamer" virus structures identified so far can be represented by closest-packing arrangements of equal morphological units composed of equal regular pentagons on a cylinder and on a sphere.

INTRODUCTION

Structural analyses show that the pentavalent and hexavalent capsomers of some icosahedral viruses are all pentamers (Rayment et al., 1982; Baker et al., 1991; Liddington et al., 1991), for which quasi-equivalent bonding specificity among the equal protein subunits suggested by Caspar and Klug (1962) is not conserved (Klug, 1983). In a recent study Marzec and Day (1993) have presented a model in which a capsomer is represented by a charge density on the surface of the sphere, and the capsid pattern is represented by superposition of such densities, with icosahedral symmetry, where the electrostatic free energy of charged capsomers is a minimum. This equilibrium model for 72 pentameric capsomers resulted in the pattern observed at papilloma (Baker et al., 1991) and SV40 (Baker et al., 1989) viruses.

Virus structures consisting of a closed capsid shell built from repeated copies of a given subunit must conform to certain geometrical requirements. They are also subject to physical constraints, if suitably stable structures are to exist. Examination of the types of shell that actually occur can therefore in principle give clues to the mathematical rules that represent the (as yet unknown, or partly known) physical constraints for building. One way of examination can be physical in nature as presented, e.g., by Marzec and Day (1993); another way can be purely geometrical. In this paper we will analyze the geometrical properties of these "all-pentamer" viruses. Modeling pentamers as regular pentagons, we make a connection between the "all-pentamer" virus structures and the mathematical problem of the densest packing of a given number of non-overlapping equal regular spherical pentagons on a sphere. It can be shown that the 72 pentamers of the icosahedrally symmetric polyoma virus capsids assembled *in vitro* from the major capsid protein VP_1 (Salunke et al., 1986, 1989) and those of SV40 (Liddington et al., 1991) do not provide the closest packing of equal pentagons on a sphere with icosahedral symmetry. For papillomaviruses, however, there is a connection between the experimentally observed structure (Baker et al., 1991) and the closest-packing arrangements of pentagons on a sphere. We show here that the arrangements of pentamers of papillomaviruses, taking concentric spherical sections of the twisted subunits at two extreme radii, correspond to two different locally optimal (closest) packings of 72 equal pentagons on a sphere under the constraint of icosahedral symmetry. We also present two locally optimal packings with octahedral symmetry of 24 equal pentagons on a sphere, from which the architecture of papillomaviruses of octahedral form can be predicted.

METHODS

The actual mathematical problem is the following. How must n equal non-overlapping regular spherical pentagons be packed on the unit sphere so that the side length of the pentagons will be as large as possible? This problem seems to be similar to the Tammes problem of closest packing of equal circles on a sphere (Fejes Tóth, 1972), but there are significant differences between the two problems. In our investigations the pentagon packing problem has been analyzed, and the particular pentagon packings have been determined, by means of structural mechanics and a variant of the "heating technique" developed for spherical circle packings (Tarnai and Gáspár, 1983), resulting in a local optimum.

In our method (Tarnai and Gáspár, 1994) the pentagons are considered as rigid bodies lying on the surface of the sphere such that their size can be changed simultaneously and in the same proportion. (One can imagine the spherical pentagons to be represented by their sides, and the sides to be represented by telescopic circular arches such that the ends of an arch are considered as pin joints that are joined by a straight bar whose length changes because of a change in its temperature. The obtained plane pentagon should be, of course, braced by two diagonal bars.) Two pentagons can

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Address reprint requests to Dr. Tibor Tarnai, Technical University of Budapest, Department of Mechanics, H-1521 Budapest, Müegyetem rkp 3, Hungary. Tel.: 36-1-463-1431; Fax: 36-1-463-1099.

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touch each other along a line segment or at a point. It is supposed that the boundaries of the pentagons can slide on each other without friction. The temperature of the bars and so the side length of the pentagons is increased uniformly while the system of pentagons remains free of stress. The side length is sought for which the pentagons just start to press each other and the system of pentagons gets to a stable state of self-stress. In this state the side length of the pentagons cannot be increased further with continuous change in the temperature. Appearance of a stable state of self-stress indicates a local optimum.

In the case of the circle packing problem, two circles can touch only in one way, and if two circles are in contact then the line of action of the contact force is uniquely determined. (It passes through the centers of the circles.) In the case of the pentagon packing problem, however, two pentagons, say A and B, can touch in three different ways.

1) A side of pentagon A and a side of pentagon B have a segment in common. In this case the direction of the contact force is unique (perpendicular to the common segment), but the locus of the line of action of the contact force is not.

2) A vertex of pentagon A is lying on a side of pentagon B, but the two pentagons have no segment of their sides in common. In this case the line of action of the contact force is uniquely determined. (It is perpendicular to the side of pentagon B and passing through the vertex of pentagon A.)

3) The two pentagons have a vertex in common, but have no sides in common. In this case the line of action of the contact force is passing through the common vertex; its direction, however, is not unique. (The line of action is lying in an angular domain determined by the sides having the vertex in question as a common point.) In contrast to types 1 and 2, this type of contact is not stable. For an arbitrary small change in the position of the pentagons, this type of contact vanishes and a contact of type 2 appears instead.

These different possibilities of contacts, and thus of the contact forces, make the pentagon packing problem much more difficult than the circle packing problem.

By increasing the temperature of the pentagons we look for an arrangement in which the system of pentagons is in a stable state of self-stress. However, no force pointing outward is allowed on a pentagon. If there is a force pointing outward, it means that one of two touching pentagons wants to depart from the other. We must not prevent this departure, so such contacts must be removed. For the modified structure we repeat the heating process and removal of contacts in tension until each contact will be in compression indicating that the side length of the pentagons has a local maximum.

In the actual computations, the heating process has been simulated. Using spherical trigonometry, we describe the equilibrium configuration with a set of equations in which, in addition to the edge length of the pentagon, some angles are the unknowns. This set of nonlinear equations is solved by multiple iteration.

For packings of pentagons on a cylinder or in the plane, the method presented here does not work. For such cases we have applied intuitive constructions. The computations have been carried out by an IBM/AT 486-compatible personal computer. The FORTRAN programs used in the computations and the PASCAL programs used in the preparation of some of the drawings have been developed by us.

PENTAGON PACKINGS ON A CYLINDER

As stated by Salunke et al. (1989, p. 898), "Geometrically, the design of the assemblies of pentameric capsomers can be described in terms of packing arrangements of regular pentagons." Two pentagons are called neighbors if their boundaries have at least one point in common. A pentagon packing is called k-neighbor packing if each pentagon has k neighbors. Equal regular pentagons can be packed regularly on a plane, as shown in Fig. 1. They can also be packed on a cylinder: if the pentagons are arranged on a helical surface lattice, they form a lattice-like packing on the unrolled cylindrical surface, i.e., in the plane. The density of a lattice-like packing of pentagons in a plane is defined as the ratio of the area of the part of the lattice unit cell covered by pentagons to the area of the unit cell.

The idealized plane pentagonal tessellation corresponding to a pentamer tube (Kiselev and Klug, 1969) of papovaviruses is a five-neighbor packing of equal regular penta-

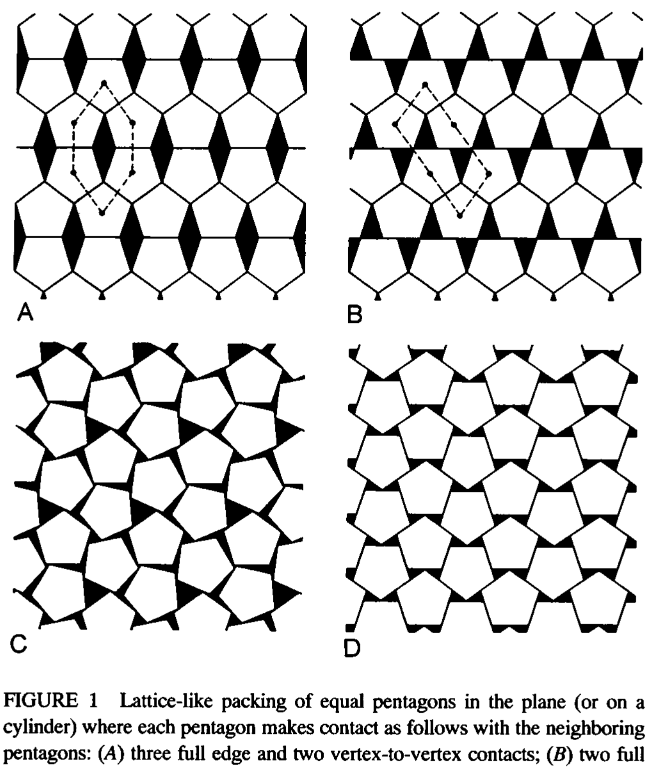

FIGURE 1 Lattice-like packing of equal pentagons in the plane (or on a cylinder) where each pentagon makes contact as follows with the neighboring pentagons: (*A*) three full edge and two vertex-to-vertex contacts; (*B*) two full edge, one edge-overlap, and three vertex-to-vertex contacts; (*C*) six edge-to-vertex contacts; and (*D*) four edge-overlap and two edge-to-vertex contacts. At each pentagon there are: (*A*, *B*) three trimer connections; (*C*) one quasi-trimer connection; (*D*) no trimer connection among pentagon vertices. *D* provides the conjectured densest packing configuration. Only *A* and *B* have been observed to date as pentamer packings in tubular variants of papovaviruses (Kiselev and Klug, 1969; Rayment et al., 1982).

gons on a $p2$ plane lattice, where each pentagon makes a full edge contact with three pentagons (Fig. 1 *A*). The six-neighbor packing of pentagons on a $p2$ plane lattice (Fig. 1 *B*) corresponding to a "hexamer" tube (Baker et al., 1983; Baker and Caspar, 1984) of polyomaviruses, which still consists of pentamers, can be obtained from the five-neighbor packing by translation of "ribbons" formed by pentagons connected by adjacent edge-to-edge contacts (each pentagon now has two full edge contacts); therefore it has the same packing density as the five-neighbor packing. However, Kiselev and Klug (1969) have noted that the packing in Fig. 1 *A* (and consequently in Fig. 1 *B*) does not represent the closest packing of pentagons in the plane. For instance, there are six-neighbor arrangements of equal pentagons on a $p3$ plane lattice (Grünbaum and Shephard, 1989), where the pentagons are more closely packed (Fig. 1 *C*). The conjectured closest packing of pentagons (Kuperberg and Kuperberg, 1990) is the $p2$ lattice six-neighbor packing shown in Fig. 1 *D*. In this arrangement there is no full edge contact of pentagons. The packing density data for these examples are collected in Table 1.

PENTAGON PACKINGS ON A SPHERE

Icosahedral assemblies

The optimal packing of pentagons on a sphere, instead of a cylinder, is much more problematic; and it is more difficult than the related problem of packing of circles on a sphere (Fejes Tóth, 1972; Tarnai and Gáspár, 1983, 1987). Investigating the "spherical" assembly of pentamers of polyomavirus capsid protein VP_1, Salunke et al. (1989) have found that equal regular pentagons representing the pentamers provide a dense packing when the number of trimer connections among pentagon corners is a maximum. (The density of packing on the sphere is defined as the ratio of the total area of the surface of the spherical pentagons to the surface area of the sphere.) Mathematically, however, maximizing the packing density and maximizing the number of trimer connections are two different conditions that can lead to different arrangements.

The arrangement of 72 equal pentagons on a sphere with the maximum number of trimer connections between the pentagon vertices is a five-neighbor packing with icosahedral symmetry (Fig. 2 *A*); but it is in an unstable state of self-stress, so its density is not a maximum—not even a local maximum. This configuration is related to the plane pentagon packing in Fig. 1 *A*. (In both configurations the centers of the pentagons determine similar elongated hexagons.) The arrangement of Fig. 2 *A* has not been observed in the pentamer packing in a virus capsid. If the clusters of six pentagons at the vertices of the icosahedron in Fig. 2 *A* are rotated about the fivefold symmetry axes, and the pentagons are increased in size (while the number of trimer connections decreases), the packing of 72 equal pentagons on a $T = 7$ icosahedral surface lattice (Fig. 2 *B*) is obtained. This corresponds to the capsid of polyomaviruses (Rayment et al., 1982; Salunke et al., 1989); but again it does not result in a maximum packing density, nor even in a local maximum. The pentagons corresponding to the hexavalent pentamers have six neighbors. The configuration in Fig. 2 *B* resembles in some respects the plane pentagon packing in Fig. 1 *B* where the elongated hexagons are skewed. On the sphere, however, if only two pentagon corners meet, they

TABLE 1 Density of packing of equal regular pentagons

Label of subfigure	Planar or cylindrical packing (Fig. 1)	72 pentagons in icosahedral packing (Fig. 2)	24 pentagons in octahedral packing (Fig. 3)
A	0.85410	0.84304	0.81071
B	0.85410	0.84813	0.85139
C	0.86465*	0.86536	0.85981
D	0.92131	0.88381	0.86795

*This packing density occurs when the largest interstitial equilateral triangle has edge length equal to 0.8 of the edge length of the pentagons, as shown in 1 *C*. As the edge length of this triangle decreases from 1.0 to 0.64311, the packing density increases from a minimum 0.84114 to a maximum 0.87048.

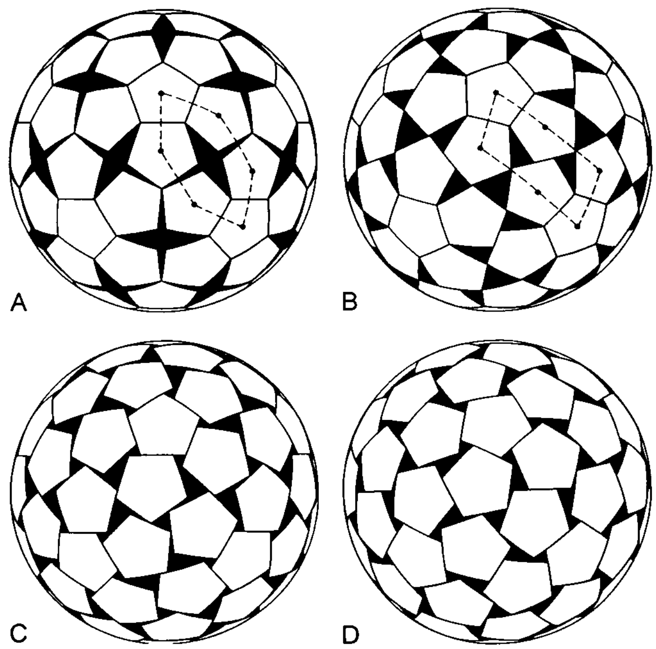

FIGURE 2 Packing of 72 equal pentagons on a sphere with icosahedral symmetry: (*A*) with a plane of symmetry, and (*B*, *C*, *D*) on a $T = 7$ surface lattice. Each pentagon at the vertices of the icosahedron makes contact as follows with the neighboring pentagons: (*A*, *B*) five full edge and (*C*, *D*) five edge-overlap contacts. Each of the other pentagons makes contact as follows with its neighbors: (*A*) one full edge and four vertex-to-vertex contacts; (*B*) one full edge, one edge-overlap, two edge-to-vertex, and two vertex-to-vertex contacts; (*C*, *D*) two edge-overlap and two edge-to-vertex contacts. In *A* (analogous to Fig. 1 *A*) there are 80 trimer connections; in *B* (analogous to Fig. 1 *B*) there are 60 trimer connections; in *C* and *D* (analogous to Fig. 1 *D*) there are no trimer connections among the pentagon vertices. *D* provides the conjectured densest packing configuration with icosahedral symmetry. To date *B*, *C*, and *D* have been observed as pentamer packings in spherical variants of papovaviruses (Salunke et al., 1989; Liddington et al., 1991; Baker et al., 1991).

form only a quasidimer connection, because the point the two pentagons have in common is not quite a vertex of both pentagons.

If the densest packing of equal pentagons with icosahedral symmetry is sought, then two locally optimal configurations are obtained on a $T = 7$ surface lattice (Fig. 2, *D* and *C*). Neither of them has trimer connections among the pentagon vertices. These two locally densest packings correspond to the arrangements of the two different cross sections of the pentamers of papillomaviruses at two different radii (Fig. 7, *E* and *C* of Baker et al., 1991), at the capsid shell where the subunits form a pentagonal shape (corresponding to Fig. 2 *D*), and above the outer surface of the capsid where the subunits form a regular five-pointed, star-shaped head (corresponding to Fig. 2 *C*). The experimental observations (Baker et al., 1991) show that each of the subunits of the pentamers makes a pronounced twist of 30° just above the outer surface of the capsid shell. For geometrically ideal pentagons the corresponding angle of rotation is 26.80°; Fig. 2 *C* is obtained from Fig. 2 *D* by a left-handed rotation of the pentagons through this angle with a change of the size of the pentagons. The packing in Fig. 2 *D* represents the conjectured closest packing with icosahedral symmetry of 72 equal regular pentagons on a sphere. (The packing density data can be seen in Table 1.) In Fig. 2, *C* and *D*, each pentagon corresponding to hexavalent pentamers of papillomaviruses has four neighbors, a situation analogous to the plane pentagon arrangement obtained from Fig. 1 *D* by a slight lateral compression under which the pentagons slide on each other and the edge-to-vertex contacts cease to exist.

Goldberg (1967) has pointed out that the structure of icosahedral viruses in general do not correspond to the solution to the problem of the densest packing of equal circles on a sphere. But if pentagons are considered instead of circles, then, in spite of the icosahedral symmetry restriction, this remains true for polyomaviruses. However, it seems that papillomaviruses do correspond to the solution to the problem of the densest packing of equal pentagons on a sphere with icosahedral symmetry. The fact that a packing analogy emerges when the featureless circle is replaced by the more "realistic" pentagon suggests that the optimization of a simple geometric model may correspond to some physical aspects of the packing of subunits.

Octahedral assemblies

Although the existence of octahedral viruses has been called into question, Salunke et al. (1989) have shown that polyomavirus capsid protein VP_1 can be assembled in a particle consisting of 24 pentamers arranged with octahedral symmetry. Geometric investigation shows that the packing of 24 equal pentagons, which maximizes the trimer connections among the pentagon corners, is composed of four clusters of six pentagons (five pentagons around a pentagon joined by a full edge). Such a packing can be obtained as a part of Fig. 2 *A*, with the same pentagon size. It has 22 trimer connections but a very low packing density. A greater density can be obtained with a decrease in the trimer connections. The density of the three-neighbor packing with octahedral symmetry in Fig. 3 *A* is greater, and the density can be increased even further if the full edge contacts of the pentagons are replaced by edge-overlap contacts. In this way the five-neighbor packing of 24 pentagons is obtained on a $T = 7$ octahedral surface lattice (Fig. 3 *B*) with no pentagons at the vertices of the octahedron. This geometric configuration corresponds to the octahedral aggregate of pentamers formed of polyomavirus capsid protein VP_1 identified by Salunke et al. (1989). The arrangement in Fig. 3 *B* has eight real trimer connections between the pentagon vertices, but it shows 24 additional quasitrimer connections also. (Quasitrimer means that three pentagon corners are very close to each other but the vertices do not coincide.) The structures in Fig. 3, *A* and *B*, do not represent a local maximum of packing density. If the densest packing of 24 equal pentagons with octahedral symmetry is sought, then two locally optimal configurations are obtained on a $T = 7$ surface lattice (Fig. 3, *C* and *D*) with no pentagons at the vertices of the octahedron. Neither of them has trimer connections among the pentagon vertices. The two

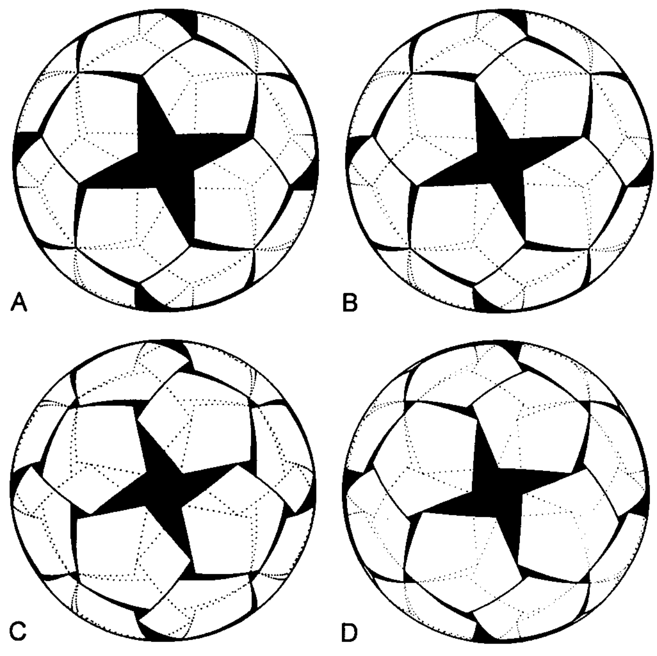

FIGURE 3 Packing of 24 equal pentagons on a sphere. (Broken lines show back of sphere.) The pentagons are situated at the points of a $T = 7$ octahedral surface lattice except at the vertices of the octahedron. Each pentagon makes contact as follows with the neighboring pentagons: (*A*) one full edge and two vertex-to-vertex contacts; (*B*) one edge-overlap, two vertex-to-vertex, and two edge-to-vertex contacts; (*C, D*) one edge-overlap and four edge-to-vertex contacts. In *A* and *B* there are eight trimer connections (in *B*, 24 additional quasi-trimer connections); in *C* and *D*, however, there are no trimer connections among the pentagon corners. Each of these packings is analogous to the similarly labeled picture in Fig. 2. *D* provides the conjectured densest packing with octahedral symmetry. Only *B* has been observed to date as pentamer packing formed in vitro from major capsid protein VP_1 (Salunke et al., 1989).

five-neighbor locally densest packings have not been observed experimentally to date as pentamer packings in virus capsids, but they can be considered as views of two concentric truncations of a predicted octahedral variant of papillomavirus capsid having twisted capsomers. Fig. 3 *C* is obtained from Fig. 3 *D* by a left rotation of the pentagons by an angle of 17.64° representing a predicted twist of the capsomers. The packing in Fig. 3 *D* provides the conjectured closest packing of 24 equal regular pentagons on a sphere with octahedral symmetry. The packing density data are collected in Table 1.

DISCUSSION

Survey of the approaches

Our primary aim was to analyze the formal mathematical problem of pentagon close packings suggested by Salunke et al. (1989). For packing of 72 equal pentagons on a sphere, we determined the locally densest configuration in Fig. 2 *C*. For years we thought this packing to be the conjectured solution to the problem. Recently, however, consulting Baker et al. (1991), we were greatly surprised not only to see this geometrical picture as an actual form of papillomavirus structure, but to see another configuration of pentamers we did not consider before. After that assembly we found another local optimum (Fig. 2 *D*) better than the previous one. Thus, pentamer virus structures could be of help to a mathematician. The different pentagon packings on a sphere presented here are new mathematical results. At the same time these new results can help the understanding of the difficult problem of pattern formation of "all-pentamer" virus structures.

For explanation of pattern formation there seem to be three approaches: the chemical approach of Caspar and Klug (1962) and of Salunke et al. (1989), which maximizes bonding ability; the physical approach of Marzec and Day (1993), which minimizes free energy; the mathematical approach of this paper, which maximizes packing density.

For icosahedral pentamer viruses the chemical approach does not seem to be suitable, because the quasi-equivalent bonding specificity suggested by Caspar and Klug (1962) is not conserved, and the chemical packing proposal of Salunke et al. (1989), that the number of trimer connections be maximized, does not work. For papillomaviruses (Baker et al., 1991), for instance, there are no trimer connections at all between pentamer corners.

The physical approach to some extent is related to the mathematical approach. Both of them result in equilibrium configurations of the pentagons, but in different quality. The relation between them is similar to that between the two extreme cases of distribution of n equal point charges on the sphere, repelling each other, due to an inverse power law potential (Melnyk et al., 1977). If the power is equal to one, then the Thomson problem is obtained (Edmundson, 1992), which is analogous to the physical approach of Marzec and Day (1993). If the power tends to infinity, then the Tammes problem of densest packing of equal circles is obtained (Kottwitz, 1991), which is analogous to our mathematical approach. For small values of n, the Thomson and Tammes problems have the same solution; but for larger values of n the two problems result in different configurations, i.e., an equilibrium configuration obtained in one problem is not in equilibrium in the other. This statement is valid also for the relation between the physical approach and the mathematical approach of pentagon packings. For example, the polyoma-SV40 type structure of Marzec and Day (1993) is an equilibrium configuration due to the physical approach, but essentially the same configuration (Fig. 2 *B*) is not in equilibrium with self-stress due to the mathematical approach (consequently, it does not represent a local maximum of packing density). For papillomaviruses both approaches result in stable equilibrium configurations, but the obtained pentagon arrangements are not exactly the same. In Fig. 6 *B* of Marzec and Day (1993), the pentagon at a vertex of the icosahedron joins the five neighboring pentagons at an angle, but in our Fig. 2 *D* it makes edge-overlap contacts with them. One of the problematic points of the physical approach is that the polyoma-SV40 type structure is resulted as a saddle point of free energy, not a local minimum. It means that the assembly is in equilibrium, but the equilibrium is not stable, although the actual polyoma or SV40 virus capsid is a stable structure.

The maximum packing density approach appears to work for papilloma, but to fail for the papova pentamer tube, the polyoma "hexamer" tube, the spherical polyoma capsid, and the octahedral aggregate of polyoma VP_1 pentamers. A question arises here: why do some experimentally observed structures correspond to maximally packed pentagons, whereas others do not? In order to give a possible and relevant answer we refer to the SV40 type structure of Marzec and Day (1993), which appears to be a saddle point of free energy. As mentioned before, a saddle point represents an unstable equilibrium, and this contradicts the fact that the actual structure is stable. This obvious contradiction is resolved by the observation of Liddington et al. (1991) that there are flexible bonding arms between pentamers, which limit their motion. Mathematically, this fact can be considered as an auxiliary condition, and minimization of free energy should be replaced by a modified problem: minimization of free energy under auxiliary conditions. It is apparent that a saddle point for the original problem can be a local minimum point for the modified problem. This idea can be applied in the mathematical approach.

Close-packing of morphological units

An auxiliary condition can limit the relative motion between two or more pentagons, or can prescribe the relative positions of two or more pentagons. In the latter case, two or more pentagons in question form a rigid morphological unit, and the extremum problem with auxiliary conditions will be the following. How must m equal non-overlapping morphological units be packed on the unit sphere so that the side length of the pentagons in the morphological units will be as large as possible? In other words, the maximum density of packing of morphological units is sought. A similar problem can be posed on a cylinder or in the plane. Let us survey some possibilities for morphological units.

The single pentagon (Fig. 4 *A*) is the simplest morphological unit, and packing of its 72 copies on a sphere with icosahedral symmetry restriction (Fig. 2, *C* and *D*) results in local maxima of packing density, and corresponds to the spherical papilloma capsid.

The next possibility would be two pentagons joining by full side (Fig. 4 *B*). These pentagon pairs do not provide local maxima of packing density in any of the configurations in Fig. 1, *A* and *B*. Therefore, this pentagon pair as morphological unit is not relevant for the observed pentamer viruses. Our conjecture is that the densest packing of these pentagon pairs in the plane is that in Fig. 5 *A*, and its density is 0.88643.

A centrally symmetric pair of pentagons having a vertex in common (Fig. 4 *C*) can be considered as a morphological unit. The conjectured densest packing of these morphological units in the plane is the configuration shown in Fig. 1 *B*, which represents a local maximum of the packing density and corresponds to the polyoma "hexamer" tube. This result is in complete agreement with the experimental observations (Baker et al., 1983) that polyomavirus "hexamer" tubes consist of paired pentamers.

Three pentagons can form clusters in many different ways. The cluster in Fig. 4 *D* as a morphological unit does not seem to work. Although it is able to produce the packing configuration in Fig. 1 *A*, the obtained packing is not unique, because "ribbons" formed by the morphological units can be translated arbitrarily without changing the packing density. The cluster of four pentagons in Fig. 4 *G* does not help either. If three pentagons are arranged in a chain, the cluster in Fig. 4 *E* is obtained. This cluster as a morphological unit is able to produce the packing arrangement in Fig. 1 *A*, but the obtained packing is not unique, since the long chains formed by the morphological units can slide on each other in a range without changing the packing density. (One of the modified packings can be seen in Fig. 5 *B*). Therefore, the morphological units in Fig. 4, *D*, *E*, and *G* are not relevant.

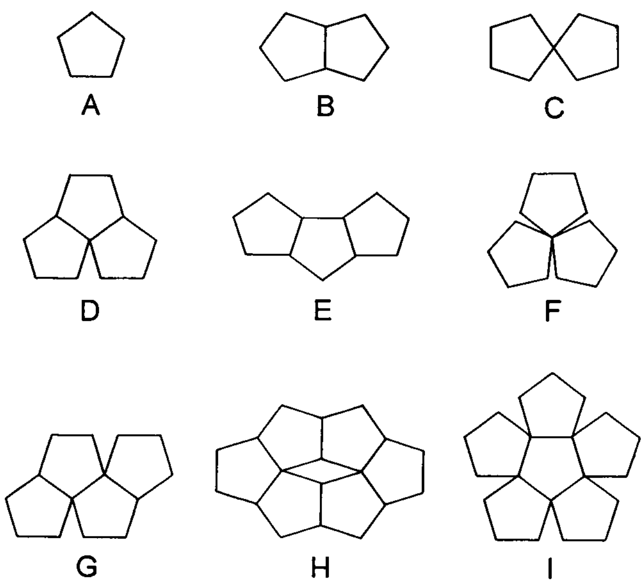

FIGURE 4 Morphological units consisting of one (*A*), two (*B*, *C*), three (*D–F*), four (*G*) and six (*H*, *I*) pentagons. The morphological units form a unique locally densest packing as follows: (*A*) in Fig. 2 C and D; (*C*) in Fig. 1 *B*; (*F*) in Fig. 3 *B*; (*H*) in Fig. 1 *A*; and (*I*) in Fig. 2 *B*.

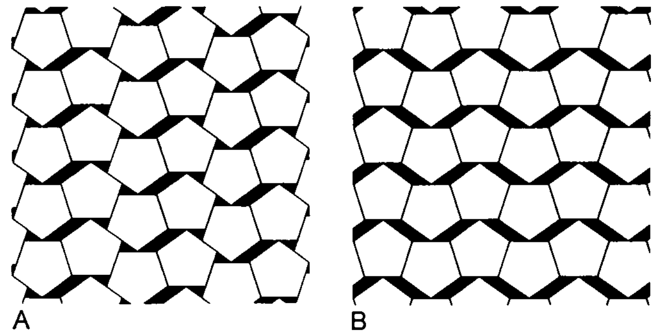

FIGURE 5 Lattice-like packings of equal morphological units in the plane (or on a cylinder). (*A*) The conjectured densest packing of the pairs of pentagons in Fig. 4 B; (*B*) a packing of pentagon triplets in Fig. 4 *E*, obtained from the packing in Fig. 1 *A* by translations of chains of pentagons.

Eight copies of morphological units in Fig. 4 *F* can be packed on a sphere as seen in Fig. 3 *B*. The packing density has a local maximum. So the mathematical approach works with the morphological unit in Fig. 4 *F*, and the obtained locally maximal packing configuration corresponds to the octahedral aggregate of polyoma VP_1 pentamers.

Considering a ring-like cluster of six pentagons (Fig. 4 *H*) as a morphological unit, we conjecture that the densest packing of these morphological units in the plane is the configuration in Fig. 1 *A*. The arrangement is unique, and its packing density has a local maximum. Thus we have a locally maximal packing of the morphological units in Fig. 4 *H* that corresponds to the papova pentamer tube.

12 identical copies of the morphological unit of six pentagons shown in Fig. 4 *I* can be packed on the sphere as seen in Fig. 2 *B*. It is an equilibrium configuration, and the packing density has a local maximum. We conjecture that this arrangement is the densest packing of the 12 morphological units on a sphere even if the icosahedral symmetry of the arrangement is not required. In this way, the obtained locally maximal packing configuration in Fig. 2 *B* corresponds to the spherical polyoma capsid; and at the same time we have a possible answer to the question, why does the spherical polyoma structure have icosahedral symmetry? In this case, however, the question to answer remains why the six pentamers assemble to form a fivefold symmetric morphological unit, and whether they do in reality. The answer to the first part of this question has been given by

Liddington et al. (1991) for SV40 virus, and they agree that the assembly might begin with a formation of a complex of five pentamers around one. However, they think that the assembly continues with the addition of individual pentamers to this complex, contrary to our suggestion.

In the mathematical approach it is also possible to imagine morphological units in which the pentagons are not rigidly attached to each other, but certain motion is allowed between them. Consider, for instance, the cluster of six pentagons in Fig. 4 *I*, and suppose that on the edges of the central pentagon, the neighboring pentagons can slide simultaneously and cyclically in the same direction and with the same distance. Our conjecture is that the locally optimal packings of 12 such morphological units are the same configurations as shown in Fig. 2, *C* and *D*, but it is not necessary to require icosahedral symmetry of packing. In this way another mathematical analogue would be obtained for papillomavirus. The hypothesis for the existence of such a morphological unit is corroborated by Fig. 7 *E* of Baker et al. (1991) based on experimental observation, where a pentavalent pentamer is connected to the neighboring hexavalent pentamers, but there is no connection between the hexavalent pentamers.

Using morphological units composed of pentagons, the mathematical approach is able to make some connection between the identified pentamer capsid structures and the locally optimal packings of morphological units, but is not able to describe the difficult intercapsomer contacts in the capsid shell. If, instead of perfect pentagons, somewhat distorted pentagons were packed on a sphere because of the combinatorially increasingly great number of different possibilities of pentagon contacts, the close packing problem would become extremely difficult. However, under some symmetry constraints, it would be possible to handle the problem. If the reference surface is not a sphere, but a somewhat flattened surface like an ellipsoid, then the mathematical approach in the presented simple form does not work. In such a case the capsomers cannot be represented by spherical pentagons but three-dimensional objects.

CONCLUSIONS

Our analysis has shown that structures formed by viruses of the polyoma/papilloma type provide an interesting insight into the formal mathematical problem of the packing of equal pentagons on a sphere and in particular show that the papillomaviruses give locally optimal solutions to the densest packing problem. At the same time the new mathematical results of the pentagon packing problem enabled us to present some conjectures and a possible mathematical (mechanical) explanation of the difficult problem of pattern formation in pentamer virus structures. It was found that all the experimentally observed structures correspond to maximally packed pentagons, or morphological units composed of two, three, or six pentagons.

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